Therefore, it is shown in the paper that the KCDS permit successful modelling of separation flows in supersonic streams of a viscous heat conductive gas.

NOTATION

$\bar{u} = (u, v)$ is the velocity; ρ is the density; p is the pressure; E is the total energy; ε is the internal energy; T is the temperature; Φ is the dissipative function; τ_{xx}, τ_{xy}, τ_{yy} are viscous stress tensor components; c is the sound speed; γ is the adiabatic index; μ, λ are the viscosity and heat conduction coefficients; Pr is the Prandtl number; Re is the Reynolds number; M is the Mach number; h is the step height; h^x, h^y are steps of the spatial grid; ω is the exponent in the Sutherland law; τ_w is friction; q is the heat flux; δ is the boundary layer thickness; n is the external normal to the surface; H is the total enthalpy. The subscripts w refers to the wall parameters, ∞ to the unperturbed stream, and 0 to stagnation.

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METHOD OF NUMERICAL SOLUTION OF ONE-DIMENSIONAL MULTIFRONT STEFAN PROBLEMS

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A finite-element method with explicit isolation of fronts is proposed for Stefan problems with an arbitrary number of phase transition boundaries.

In conformity with practical requirement for the mathematical modelling of phase transitions it is necessary to perform explicit separation of the moving boundaries in the majority of cases, that not every numerical method permits doing. For instance, algorithms based on an enthalpy formulation [1, 2] or "blurring of the front" [3] that satisfactorily determine the temperature field, yield only a rough estimate of the phase front location. Numerous literature, for which a brief survey is contained in [4, 5], say is devoted to methods with explicit separation of fronts. Recently, several different modifications of application of the finite element method with deformable computational grids has been proposed in the direction of developing this approach for the solution of problems in domains with nonstationary boundaries [6-8]. The common disadvantage of both these and other finite-element methods proposed earlier is that different numerical schemes describe heat transmission within a single-phase domain and the front motion law, where the order of accuracy of the second schemes (in direct proximity of the front) is lower, as a rule, then for the first. Besides the degradation of the accuracy, the heterogeneity of the computation schemes results in complication of the algorithm for the solution, that grows strongly as the number of fronts increases.

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On the other hand, application of known numerical methods is limited to simple Stefan problems that are of interest just as model examples while the numerical solution of more complex complex problems of practical nature is exposed slightly. This refers to the greatest degree to processes characterizing the presence of several fronts (phases) that originate and vanish in the general case. Formulation of a number of such problems in application to permafrost science and geology is contained in [9], however, their consideration is limited, in the main, to obtaining and investigating self-similar solutions. The numerical solution of a three-phase Stefan problem (vapor–water–ice) with two translationally moving fronts, phase origination and degeneration, is obtained in [7].

A new approach to the numerical solution of one-dimensional problems of Stefan type is proposed in this paper. The computational grid being used contains movable, in addition to fixed nodes, that correspond to points of a domain with a given temperature. However, the equations for each node are derived on the basis of a single integral identity. Hence, a system of difference equations is obtained that is homogeneous in structure, with different coefficients for the unknowns depending on the kind of node. The algorithm of the solution is simplified significantly while certain algorithmic complications are injected only by the necessary check on the origination or disappearance of the fronts being tracked. An increase in the number of fronts (phases) does not here complicate the scheme of the solution. The computational scheme remains unchanged even in the case of modelling phase transformations in multilayered media. The proposed method is conservative and its accuracy and efficiency are illustrated by numerical examples.

1. THE INTEGRAL IDENTITY

Let heat transfer in a medium be described by the one-dimensional heat conduction equation

$$C(x,\ T)\frac{\partial T}{\partial t} = \frac{1}{A(x)}\frac{\partial}{\partial x}\left(A(x)\lambda(x,\ T)\frac{\partial T}{\partial x}\right), \tag{1}$$

in which $A(x) = x^k$ ($k = 0, 1, 2$ for the appropriate kind of symmetry).

Moreover, the energy conservation law, whose expression within the framework of the circle of problems under consideration is represented in the form

$$-\lambda(x,\ T)\frac{\partial T}{\partial x}\Big|_{x=z-0} + \lambda(x,\ T)\frac{\partial T}{\partial x}\Big|_{x=z+0} = L(z,\ T)\frac{dz}{dt}. \tag{2}$$

is satisfied at any point $x = z(t)$ (moving or fixed). Depending on the nature of the point $z(t)$, equation (2) expresses either continuity of the heat flux during the passage through a point of the domain ($L(z, T) = 0$), including the condition on the contact of two media, or the known Stefan condition on a phase transition front. In the latter case $z(t)$ corresponds to a moving front on which $T(z(t), t) = T_m$, the phase transition temperature and $L(z, T_m)$ equals the latent heat of this phase transition.

The integral identity is derived by analogy with that which has been done in [10].

Let $x_{n-1}(t) < x_n(t) < x_{n+1}(t)$ be three arbitrary points of a domain. Let us introduce the functions

$$u_n(x) = \int_{x_n}^{x}\frac{d\sigma}{A(\sigma)},\quad v_n(x) = \int_{x}^{x_{n+1}}\frac{d\sigma}{A(\sigma)}.$$

Multiplication of the heat conduction equation (1) by $A(x)u_{n-1}(x)$ and integration over the domain $x_{n-1} \le x \le x_n$ yield the identity

$$\frac{1}{\Delta_{n-1}}\int_{x_{n-1}}^{x_n} C(x,\ T)\frac{\partial T}{\partial t}u_{n-1}(x)A(x)\,dx = A_n\lambda(x,\ T)\frac{\partial T}{\partial x}\Big|_{x=x_n-0} - \frac{1}{\Delta_{n-1}}\int_{x_{n-1}}^{x_n}\lambda(x,\ T)\frac{\partial T}{\partial x}\,dx, \tag{3}$$

in which the right side is obtained by taking the appropriate integral by parts and the notation

$$A_n = A(x_n),\ \Delta_n = \int_{x_n}^{x_{n+1}} \frac{d\sigma}{A(\sigma)} = u_n(x_{n+1}) = v_n(x_n).$$

is introduced to simplify the writing.

An analogous identity is obtained as a result of multiplying (1) by $A(x)v_n(x)$ and integrating over the domain $x_n \le x \le x_{n+1}$:

$$\frac{1}{\Delta_n}\int_{x_n}^{x_{n+1}} C(x,\ T)\frac{\partial T}{\partial t} v_n(x)\,A(x)\,dx = -A_n\lambda(x,\ T)\left.\frac{\partial T}{\partial x}\right|_{x=x_n+0} + \\ + \frac{1}{\Delta_n}\int_{x_n}^{x_{n+1}} \lambda(x,\ T)\frac{\partial T}{\partial x}\,dx. \tag{4}$$

Combining (3) and (4), taking account of the energy conservation condition (2) for the point $x = x_n(t)$, we finally obtain

$$\frac{1}{\Delta_{n-1}}\int_{x_{n-1}}^{x_n} C(x,\ T)\frac{\partial T}{\partial t}\,u_{n-1}(x)\,A(x)\,dx + \\ + \frac{1}{\Delta_n}\int_{x_n}^{x_{n+1}} C(x,\ T)\frac{\partial T}{\partial t}\,v_n(x)\,A(x)\,dx + A_nL_n\frac{dx_n}{dt} = \\ = \frac{1}{\Delta_n}\int_{x_n}^{x_{n+1}} \lambda(x,\ T)\frac{\partial T}{\partial x}\,dx - \frac{1}{\Delta_{n-1}}\int_{x_{n-1}}^{x_n} \lambda(x,\ T)\frac{\partial T}{\partial x}\,dx, \tag{5}$$

where $L_n = L(x_n,\ T(x_n))$. The identity (5) is valid for any interior point $x_n(t)$ of the domain under investigation, at the same time the identities (3) and (4) are valid for its boundary points.

2. THE NUMERICAL SCHEME

Let us give a computational grid represented by a set of nodes of two kinds in the domain under investigation. The ordinary fixed nodes given by their coordinates and distributed in a space with a certain nonuniform, generally speaking, step comprise one group. The temperature therein is unknown and determine during the numerical solution. Nodes of the second kind are given by a set of values in front of the given temperatures (isotherms) at the moving nodes whose coordinates are to be determined. In particular, the nodes of such kind are needed for modelling the phase transition boundaries although that is not the limit to their usefulness. In a number of cases the application of node-isotherms is more preferable since it yields a more optimal distribution of computational grid nodes and thereby permits reducing their number without loss of accuracy. This advantage appears most obviously in the solution of problems in unbounded domains.

In contrast to the fixed nodes, the moving modes can occur and vanish as time lapses, where isotherm origination in the absence of distributed heat sources is possible only from the domain boundaries. Finally, for the approach described the phase transition front is represented by a node-isotherm with an appropriate non-zero value of the latent heat (see (2) and (5)).

We approximate the temperature in the segment $x_n \le x \le x_{n+1}$ bounded by two successive nodes (outside the dependence on their type) by a pseudostationary approximation

$$T(x,\ t) = \psi_n\frac{v_n(x)}{\Delta_n} + \psi_{n+1}\frac{u_n(x)}{\Delta_n}, \tag{6}$$

TABLE 1. Expressions for the Coefficients of Eqs. (9) and (10) Depending on the Kind of Node

ζ	a_n	b_n	d_n	c_n	z_n
ψ	$z_{n-1}W_{n-1}$	$z_{n-1}U_{n-1}$	z_nV_n	z_nW_n	$\frac{C_n}{(\Delta_n)^2}$
x	$\frac{z_{n-1}}{A_{n-1}}W_{n-1}$	$\frac{z_{n-1}}{A_n}U_{n-1}$	$\frac{z_n}{A_n}V_n$	$\frac{z_n}{A_{n+1}}W_n$	$\frac{C_n\Delta\psi_n}{(\Delta_n)^3}$

$$U_n=\int_{x_n}^{x_{n+1}}[u_n(x)]^2A(x)\,dx,\quad V_n=\int_{x_n}^{x_{n+1}}[v_n(x)]^2A(x)\,dx,$$

$$W_n=\int_{x_n}^{x_{n+1}}u_n(x)\,v_n(x)\,A(x)\,dx$$

such that T(x, t) satisfies the stationary heat conduction equation for a given symmetry in the fixed domain $x_n \le x \le x_{n+1}$ for $\lambda(x, T) = \text{const}$ as well as the conditions $T(x_k, t) = \psi_k$, $k = n, n+1$, where ψ_k denotes the temperature at a node with the coordinate x_k.

Differentiating (6) with respect to the time yields

$$\frac{\partial T}{\partial t}=\dot{\psi}_n\frac{v_n(x)}{\Delta_n}+\dot{\psi}_{n+1}\frac{u_n(x)}{\Delta_n}+\dot{x}_n\frac{v_n(x)\Delta\psi_n}{A_n(\Delta_n)^2}+\dot{x}_{n+1}\frac{u_n(x)\Delta\psi_n}{A_{n+1}(\Delta_n)^2}, \tag{7}$$

where $\Delta\psi_n=\psi_n-\psi_{n+1}$ and either ψ_n, or $\dot{x}_n$ equals zero for each n depending on the kind of n-th node. Letting ζ_n denote an unknown quantity at the n-th node that equals ψ_n for an isothermic node, we reduced (7) to the form

$$\frac{\partial T}{\partial t}=\beta_n\dot{\xi}_n+\gamma_n\dot{\xi}_{n+1} \tag{8}$$

with appropriate expressions for the coefficients β_n, γ_n from (7).

The equations for the numerical solution are derived by substituting the approximation (8) into the appropriate identity (3)-(5). We here set $C_n=(C(x_n,\ \psi_n)+C(x_{n+1},\ \psi_{n+1}))/2$, $\lambda_n=(\lambda(x_n,\ \psi_n)+\lambda(x_{n+1},\ \psi_{n+1}))/2$ in each interval $x_n \le x \le x_{n+1}$.

For the internal nodes we obtain the following equations from (5):

$$a_n\dot{\xi}_{n-1}+(b_n+d_n+e_n)\dot{\xi}_n+c_n\dot{\xi}_{n+1}=f_n,\ n=2,\ \ldots,\ N-1, \tag{9}$$

where $e_n=A_nL_n$; $f_n=\lambda_n(\psi_{n+1}-\psi_n)/\Delta_n-\lambda_{n-1}(\psi_n-\psi_{n-1})/\Delta_{n-1}$, while the expressions for the remaining coefficients of the unknowns are presented in Table 1, depending on the kind of node.

Equations for the boundary nodes can be obtained analogously from (3) or (4). Thus, upon giving a known function of the time Q(t) at the left boundary of the heat flux for the node n = 1, we will have from (4)

$$(e_1+d_1)\dot{\xi}_1+c_1\dot{\xi}_2=f_1,\ f_1=\lambda_1(\psi_2-\psi_1)/\Delta_1+A_1Q(t), \tag{10}$$

where (10) is valid for both a fixed boundary of the domain and for an external phase front that occurs during ablation for instance

TABLE 2. Coordinates of the Paraffin Solidification From x_s and Thawing Front x_m for the Self-Similar Problem

t, days	20	40	60	80	100
x_s, м	—0,3870 —0,3934	—0,5506 —0,5564	—0,6766 —0,6814	—0,7829 —0,7868	—0,8767 —0,8797
x_m, м	0,7189 0,7245	1,0184 1,0246	1,2481 1,2549	1,4416 1,4490	1,6119 1,6201

Note. The upper value corresponds to the numerical solution and the lower to the exact solution.

The conditions of the contact of two media are taken into account automatically during the derivation of (9).

Independently of the kind of boundary conditions, the nmber of phase fronts and the medium being multilayered, the approximate solution will be determined by a system of ordinary differential equations with tridiagonal coefficient matrices

$$A(\bar{\zeta})\dot{\bar{\zeta}} = \bar{f}(\bar{\zeta}), \qquad \bar{\zeta} = (\zeta_1, \ldots, \zeta_N)^T, \ \bar{f} = (f_1, \ldots, f_N)^T. \tag{11}$$

The simplest implicit discretization with respect to time reduces (11) to a system of nonlinear difference equation

$$F(\bar{\zeta}) = A(\bar{\zeta})(\bar{\zeta} - \bar{\zeta}^m) - \Delta t \bar{f}(\bar{\zeta}) = \bar{0}, \tag{12}$$

where $\bar{\zeta} = \bar{\zeta}^{m+1}$; $\bar{\zeta}^m = \bar{\zeta}(m\Delta t)$.

The system (12) is solved effectively by the Newton method by neglecting variations of the components of the matrix $A(\bar{\zeta})$ within one iteration so that each iteration step requires the solution of the linear system

$$D(\bar{\zeta}^\nu)(\bar{\zeta}^{\nu+1} - \bar{\zeta}^\nu) = -F(\bar{\zeta}^\nu), \tag{13}$$

where $D(\bar{\zeta}) = A(\bar{\zeta}) - \Delta t J(\bar{\zeta})$ and $J(\bar{\zeta}) = (\partial f_i/\partial \zeta_j)_{i,\, j=\overline{1,N}}$ is the Jacobi matrix of the vector function $\bar{f}$ and ν is the number of the iteration. Exactly as A the matrix J is tridiagonal so that an ordinary factorization method is used for (13). The components of the matrix J are easily evaluated explicitly. Experiments performed showed a good convergence of the numerical procedure (13) that requires not more than 2-4 iterations, as a rule, for satisfaction of the conditions $|\zeta_n^{\nu+1} - \zeta_n^\nu| < 10^{-5}$. Let us note that the simple iteration method is not effective for the solution of the system (12).

The implicit difference scheme (12) guarantees stability of the method independently of the magnitude of the time step, which is of value, in principle, in the conditions of the moving computational grid.

Let us note that derivation of the system (11) for an approximate solution of the circle of problem under consideration is analogous to the procedure for constructing numerical schemes in the finite element method. In particular, utilization of one system of functions as basis and weight functions ($\{u_n(x), v_n(x)\}$ in this method) is characteristic for the Galerkin method while integration by parts to eliminate gradient terms corresponds to a weak formulation of the Galerkin method [11]. Therefore, the numerical scheme proposed is a finite-element scheme and can be obtained formally from the aspect of the finite-element method (FEM) by taking account of the computational grid structure used. The difference between the proposed and the known finite-element methods of solving Stefan type problems in this sense is the most complete utilization of FEM possiblities for construction of numerical equations for the unknown moving boundaries.

In the particular case when the computational grid is represented just by moving nodes with a fixed temperature, the numerical solution reduces to tracking the isotherm motion.

Such an approach was first proposed in [12] for the numerical solution of heat conduction problems. The method elucidated contains a realization of the idea of tracking the isotherm on the basis of an implicit finite-element scheme.

3. NUMERICAL EXAMPLES

The method was realized in the form of a FORTRAN language program for the ES electronic computer and was approved in different problems. The numerical solution of a contact two-front Stefan problem that is of interest from both the viewpoint of the numerical solution and for practical applications is presented as an illustration.

Let two bodies having different thermophysical characteristics and different initial temperatures T_ℓ^0 and $T_f^0 < T_\ell^0$ (for simplicity we assume T_ℓ^0, T_f^0 = const), be set instantaneously into ideal contact with each other. For definiteness, we will consider the first body to be in the liquid phase with the solidification temperature T_s^* and the second in the solid phase with the melting point T_m^*. At the initial instant of contact t = 0, a certain temperature $T_0(0)$ is set on the boundary of the two bodies such that $T_\ell^0 > T_0(0) > T_f^0$. Depending on the quantity $T_0(0)$ the thermal interaction of the bodies during contact can initiate a phase transition in one of them or in both simultaneously upon the satisfaction of the conditions $T_\ell^0 \geq T_s^* > T_0(0) > T_m^* \geq T_f^0$. For specific cases the determination of $T_0(0)$ and the conditions for origination of the phase transitions can be satisfied on the basis of a self-similar solution of the appropriate contact problem for two semi-bounded bodies.

The origination of two phase transitions occurs simultaneously for example, upon the formation of intrusives when magma crystallization is accompanied by melting of the surrounding rock [9, 13], or upon filling a forehole in a frozen rock interval with melted paraffin. The numerical examples presented here correspond to this latter case.

The problem under consideration is described in a one-dimensional formulation by the heat conduction equations for melted and solidifying parts of paraffin and thawing and frozen soil:

$$\frac{\partial T_l}{\partial t} = \alpha_l \Delta T_l,\ a < x < x_s;\quad \frac{\partial T_s}{\partial t} = \alpha_s \Delta T_s,\ x_s < x < c;$$
$$\frac{\partial T_m}{\partial t} = \alpha_m \Delta T_m,\quad c < x < x_m;\quad \frac{\partial T_f}{\partial t} = \alpha_f \Delta T_f,\ x_m < x < b \tag{14}$$

with the boundary and initial conditions

$$x = a : A(x)\frac{\partial T_l}{\partial x} = 0;\quad x = b : A(x)\frac{\partial T_f}{\partial x} = 0;$$
$$x = c : \lambda_s \frac{\partial T_s}{\partial x} = \lambda_m \frac{\partial T_m}{\partial x},\ T_s = T_m;$$
$$t = 0 : T_l(x,\ 0) = T_l^0,\ a < x < c,\ T_f(x,\ 0) = T_f^0,\ c < x < b; \tag{15}$$
$$x_s(0) = x_m(0) = c$$

and conditions on the boundaries of paraffin solidification and thawing frozen rock

$$x = x_s : -\lambda_l \frac{\partial T_l}{\partial x} + \lambda_s \frac{\partial T_s}{\partial x} = L_s \frac{dx_s}{dt},\ T_l = T_s = T_s^*;$$
$$x = x_m : -\lambda_m \frac{\partial T_m}{\partial x} + \lambda_f \frac{\partial T_f}{\partial x} = L_m \frac{dx_m}{dt},\ T_m = T_f = T_m^*. \tag{16}$$

In the case of plane-parallel heat flow (A(x) = 1) as $a \to -\infty$, $b \to \infty$, c = 0 the problem (14)-(16) has an exact self-similar solution $x_s = -2a_s\sqrt{t}$, $x_m = 2a_m\sqrt{t}$, $T_s(0, t) = T_m(0, t) = T_0$. For the thermophysical properties taken for the paraffin λ_s = 0.23 BT/(m·K), λ_t =0.14 W/(m·K), $\alpha_s = 0.144 \cdot 10^{-6}$ m²/c, $\alpha_\ell = 0.088 \cdot 10^{-6}$ m²/c, $L_s = 116.8 \cdot 10^6$ J/m³, $T_s^* = 45°C$, the frozen soil λ_m = 1.75 W/(m·K), λ_f = 2 W/(m·K), $\alpha_m = 0.6 \cdot 10^{-6}$ m²/c, $\alpha_f = 0.91 \cdot 10^{-6}$ m²/c, $L_m = 61 \cdot 10^6$

J/m^3 (taking account of the 20% icines of the rock), $T_m^* = 0°C$ and the initial temperatures $T_m^* = 0°C$ and constants governing the self-similar solution have the following values: $a_s = 0.152748 \cdot 10^{-3}$, $a_m = 0.281305 \cdot 10^{-3}$, $T_0 = 8.83924$.

Let us first estimate the accuracy of the method by solving the problem (14)-(16) in the self-similar formulation. Computational grids of different structure were used for a more complete illustration of the method in different domains. The cooling and solidification of paraffin were investigated in the segment $-1 \leq v \leq 0$ by using a fixed grid with the step $\Delta x = 0.1$, supplemented by a moving node $\psi = 45°C$, corresponding to the solidification front. In order not to introduce additional errors associated with the replacement of the infinite by a finite domain, the temperature from the exact solution of the problem was given on the outer boundary $x = -1$. A grid of moving isotherms with a temperature step $\Delta\psi = 2.5°C$ was used for the numerical solution in the domain $0 < x < \infty$. The boundary isotherm $\psi_N = -5°C$ separates the domain $x \geq x_N(t)$ that is not subjected to thermal perturbation and has a temperature equal to the initial one at each instant. In conformity with this, the boundary condition $\partial T/\partial x = 0$ is given on the moving boundary $x = x_N(t)$.

As is seen, application of a grid of moving isotherms without additional forces permits solving the problem under consideration numerically in unbounded domains. The thawing front corresponds to the isotherm $\psi = 0°C$. The results of the numerical solution obtained for a $\Delta t \approx 1$ day time step are presented in Table 2 and Fig. 1. A comparison shows that the relative error of the solution obtained on a sufficiently rough grid does not exceed 2% in the domain $-1 \leq x \leq 0$ and 1% in the domain $0 \leq x < \infty$.

Let us examine the same problem in another formulation. Let the paraffin be in a circular cylinder of unit radius in the neighborhood of an unbounded frozen rock massif. The mathematical formulation is given by the problem (14)-(16) for plane-radial symmetry ($A(x) = x$) for $a = 0$, $b \to \infty$, $c = 1$. For the same initial data as above, solidification of the paraffin and thawing of the frozen rock start simultaneously at the time of contact. However, by virtue of the boundedness of the heat reserve of the paraffin, as it is exhausted the frozen rock thawing is stopped and replaced by the reverse process [14]. In contrast to the previous example, the water-ice front motion will be non-monotonic in nature while the temperature on the paraffin–soil contact changes with time.

Application of the isothermal grid in the $1 < x < \infty$ domain is not logical in this case since a sharp drop in the temperature at the contact specifies rapid isotherm motion during their return in the neighborhood of the contact boundary and its associated necessity to execute computations with the very smallest time step, as follows from the computations presented. Consequently, the numerical solution for this formulation was obtained on a fixed grid (with step $\Delta x = 0.1$ in domain $0 \leq x \leq 1$ and $\Delta x = 0.2$ for $1 < x \leq \ell$; where ℓ iw the boundary of the computational domain), supplemented by moving nodes $\psi = 45$ and $0°C$. The time step was taken equal to $\Delta t \approx 1$ day. The following approximation for the temperature was given on the outer boundary of the computational domain $x = 1$.

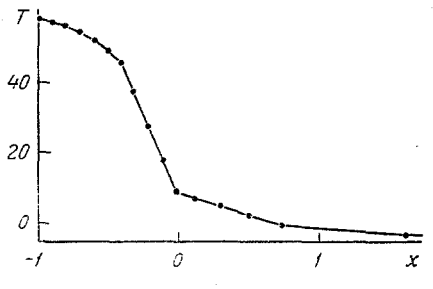

Fig. 1

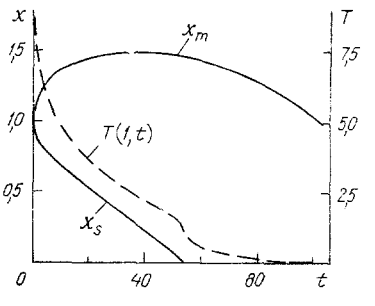

Fig. 2

Fig. 1. Temperature distribution at the time t = 20 days for the self-similar problem: solid line is the exact solution and points are the numerical solution x, m; T, °C.

Fig. 2. Dynamics of phase boundary motion and change in temperature at the parafin-soil contact for the plane-radial case, t in days.

$$T(l,\ t) = -\frac{\ln l/x_m}{\ln\left[1+\dfrac{l-x_m}{x_m(1-u)}\right]},\quad u = \operatorname{erfc}\frac{l-x_m}{2\sqrt{t/\alpha_f}}.$$

Assignment of such a condition on the outer boundary for the numerical solution of Stefan problems in an unbounded domain, that was proposed and approved by numerical experiments in [15], permits a more significant (more than twice in this example) diminution in the size of the computational domain than for assignment of the condition $\partial T/\partial x = 0$ or $T(\ell, t) = T(\ell, 0)$ and can be recommended to reduce the computational expenditures. In the problem under consideration it is sufficient to take $\ell = 4$ to eliminate errors due to passage over to a finite computational domain. Certain results of the numerical solution, illustrating the dynamics of phase front motion and the change in temperature at the contact, are presented in Fig. 2.

NOTATION

x is a coordinate, t is time; T is temperature; α is thermal diffusivity; λ is the heat conduction coefficient; C is the volume specific heat; L is the heat of phase transition; ψ is the nodal temperature; ζ is a generalized variable at the node; Δt is the time step; N is the number of computational grid nodes; Δ is the Laplace operator. Subscripts: n is for the number of computational grid nodes; ℓ is for liquid paraffin; s is for solid paraffin; m is for thawed soil; f is for frozen soil.

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